On linear programming and robust model-predictive control using impulse-responses

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Abstract Campo and Morari have derived a linear programming problem, with a potentially large number of constraints, which is equivalent to a min–max formulation for robust model-predictive control of linear systems. That formulation involves minimization, with respect to the controls, of the maximum, with respect to the system’s impulse response (from a set of possible impulse responses), of the infinity norm of the error between the predicted and required system output sequences. Here an alternative linear programming problem is derived which has a smaller number of constraints and is therefore potentially more convenient for on-line control.

Keywords: Model predictive control; robust control; digital control; linear programming.

1. Introduction

Interest is increasing in the application of Model Predictive Control (MPC) to industrial process control, mainly because of its simplicity and easy handling of constraints. The recent rapid increase in cheap computing power has contributed to its practicability.

The robust MPC strategy considered here involves minimization with respect to the control sequence of the maximum (with respect to the impulse response as it ranges over a pre-specified set) of the infinity norm of the error between the predicted output sequence and the demanded output sequence. That formulation was converted into a potentially large linear programming problem in [1]. A similar formulation and linear programming problem are described in Section 2.

The contribution of this paper is the derivation in Section 3 of an equivalent linear programming problem which has a smaller, and usually a much smaller, number of constraints.

2. A linear programming formulation of a min–max problem

At the present discrete time $t^k$, the future control sequence, extending to the (receding) horizon of interest at $t^{k+M}$, is specified by $u = (u^k, u^{k+1}, \ldots, u^{k+M-1})' \in R^{mM}$ where $u^i \in R^m$ for each $i$. It is assumed that only the past $N$ control values $u^i$ can affect the present and future outputs. Hence the relevant initial condition for the optimization interval is specified by the known vector $v = (u^{k-1}, u^{k-2}, \ldots, u^{k-N})' \in R^{mN}$.

The output sequence corresponding to the concatenation of $v$ and $u$ is denoted by

$$y(H_\xi, u, v) := (y^{k+1}(H_\xi, u, v)', y^{k+2}(H_\xi, u, v)', \ldots, y^{k+M}(H_\xi, u, v)')' \in R^{rM}$$

with each $y'(H_\xi, u, v) \in R^r$. Here, $H_\xi := (H^1_\xi, H^2_\xi, \ldots, H^N_\xi)$ specifies the impulse response sequence for the system considered, where each $H^i_\xi \in R^{r \times m}$ and

$$y^{k+i}(H_\xi, u, v) = \sum_{i=1}^{N} H^i_\xi u^{k+i-1}.$$
Uncertainty in the impulse response is modelled by having \( H_i = \sum_{j=1}^{N} H_j \xi_j \) where \( H_j \in \mathbb{R}^{m \times m} \) and all that is known about the unknown \( \xi \) is that it lies in the set \( \Xi = \{ \xi \in \mathbb{R}^q : \xi_1 \leq \xi \leq \xi_N, \forall t \in \langle q \rangle \} \). Here \( \langle q \rangle = \{1, 2, \ldots, q\} \) and the bound vectors \( \xi_1, \xi_N \in \mathbb{R}^q \) are pre-specified.

As in [1], the associated optimization problem can be formulated as that of minimizing, with respect to \( u \), the infinity norm of the worst case, with respect to \( s \), of the error between the predicted output sequence \( y(H_i, u, v) \) and the desired output sequence \( s = (s_{k+1}, s_{k+2}, \ldots, s_{k+M})' \). The corresponding optimization problem is for given \( v \),

\[
\begin{align*}
\text{(P1)} \quad \min_{u} & \quad \max_{\xi \in \Xi} \| y(H_i, u, v) - s \|_{\infty}, \\
\text{subject to} & \quad u \in \mathbb{R}^{mM}, \quad u^L \leq u \leq u^U, \\
& \quad y^L \leq y(H_i, u, v) \leq y^U, \forall \xi \in \Xi,
\end{align*}
\]

where the bound vectors \( u^L, u^U \in \mathbb{R}^{mM} \) and \( y^L, y^U \in \mathbb{R}^{rM} \) are pre-specified.

Let \( \tilde{\Xi} := \{ \xi \in \mathbb{R}^q : \xi_i \in \{\xi_1^L, \xi_1^U\}, \forall i \in \langle q \rangle \} \) and let \( \{\xi_i : i \in \langle q \rangle\} \) be the values of \( \xi \) which constitute \( \tilde{\Xi} \). The \( \xi_i \) are the extreme points of \( \Xi \) so \( \Xi \) is the convex hull of the \( \xi_i \). Consequently, since \( y(H_i, u, v) \) is a linear function of \( \xi \) for given \( (u, v) \),

\[
\begin{align*}
\{ u \in \mathbb{R}^{mM} : & \quad u^L \leq u \leq u^U, \quad y^L \leq y(H_i, u, v) \leq y^U, \forall \xi \in \Xi \} \\
= \{ u \in \mathbb{R}^{mM} : & \quad u^L \leq u \leq u^U, \quad y^L \leq y(H_i, u, v) \leq y^U, \forall \xi \in \langle 2^q \rangle \}.
\end{align*}
\]

Further, again since \( y(H_i, u, v) \) is linear in \( \xi \) for each \( (u, v) \), a maximizer \( \hat{\xi} = (\hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_N) \) for \( \Pi_1 \) is an extreme point of \( \Xi \) (in that each \( \hat{\xi}_i \) is either \( \xi_i^L \) or \( \xi_i^U \)) [1; 2, Theorem 6.4.3]. Consequently \( \Pi_1 \) is equivalent to

\[
\begin{align*}
\text{(P2)} \quad \min_{u, \xi} & \quad \max_{\xi} \| y(H_i, u, v) - s \|_{\infty}, \\
\text{subject to} & \quad u \in \mathbb{R}^{mM}, \quad u^L \leq u \leq u^U, \\
& \quad y^L \leq y(H_i, u, v) \leq y^U, \forall \xi \in \langle 2^q \rangle,
\end{align*}
\]

in the sense that both problems have the same min–max value and the solution of either can be found from the solution of the other.

Since \( \| z \|_{\infty} = \min_{\theta \in \mathbb{R}} \{ -\theta \leq z \leq \theta, \forall i \in \langle q \rangle \} \) when \( z \in \mathbb{R}^q \), it follows that problem \( \Pi_2 \) can be reformulated as the following linear programming problem:

\[
\begin{align*}
\text{(P3)} \quad \min_{\theta} & \quad \theta, \\
\text{subject to} & \quad u \in \mathbb{R}^{mM}, \quad u^L \leq u \leq u^U, \\
& \quad y^L \leq y(H_i, u, v) \leq y^U, \forall i \in \langle 2^q \rangle, \\
& \quad \theta \leq (y(H_i, u, v) - s), \leq \theta, \forall i \in \langle 2^q \rangle, \forall j \in \langle M \rangle.
\end{align*}
\]

\( \Pi_3 \) is, modulo notation, the version for the uncertainty set \( \Xi \) considered here of the linear programming problem derived in [1]. Unfortunately \( \Pi_3 \) has \( \mathcal{C}_3 = 2(m + 2q + 1)M \) constraints, which rapidly becomes a rather large number of constraints as the length \( M \) of the optimization interval and the number \( q \) of degrees of freedom for \( H_i \) increases.

It will be shown in Section 3 that \( \Pi_1 \) is also equivalent to a linear programming problem with a much smaller number of constraints, which renders the solution of \( \Pi_1 \) more practicable for non-small \( q \) and \( M \).

3. Reformulation of the min–max problem as an alternative linear program

The number, \( \mathcal{C}_3 \), of constraints for linear programming problem \( \Pi_3 \) increases rapidly with \( q \) owing to the presence of the term \( 2^q \). That term arises because linear program \( \Pi_3 \) is obtained by considering each
of the $2^q$ extreme points of $\Xi$ in order to maximize $\|y(H_\xi, u, v) - s\|_\infty$ with respect to $\xi \in \Xi$. It will be shown first that there is a more efficient way to carry out that maximization.

Clearly,

$$\max_{\xi \in \Xi} \|y(H_\xi, u, v) - s\|_\infty = \max_{\xi \in \Xi} \max_{p \in \langle rM \rangle} \max_{l \in \langle rM \rangle} \left( \frac{y^{k+1}(H_\xi, u, v) - s^{k+1}}{p} \right)$$

$$= \max_{\xi \in \Xi} \max_{p \in \langle rM \rangle} \max_{l \in \langle rM \rangle} \left( \frac{y^{k+1}(H_\xi, u, v) - s^{k+1}}{p} \right)$$

$$= \max_{\xi \in \Xi} \max_{p \in \langle rM \rangle} \max_{l \in \langle rM \rangle} \left( \frac{\sum_{i=1}^N (H_{\xi}^{k+1} - s)^p}{p} \right) \tag{3.1}$$

Let $\eta := \frac{1}{2}(\xi^U + \xi^L)$, so that $\eta$ is at the centre of the uncertainty interval $[\xi^L, \xi^U]$ for $\xi$, and let $\Delta := \frac{1}{2}(\xi^U - \xi^L)$. Then $\Delta \geq 0$ and

$$\left[ \xi^L \leq \xi \leq \xi^U \right] \iff \left[ \xi = \eta + \delta \text{ for some } \delta \in \mathcal{D} \right] \tag{3.2}$$

where $\mathcal{D} := \{ \delta \in \mathbb{R}^q : \|\delta\|_1 \leq \Delta, \forall i \in \langle q \rangle \}$. Consequently, for all $\xi \in \Xi$,

$$H_\xi = \sum_{j=1}^q H_j^{\xi} = H_\eta^k + \sum_{j=1}^q H_j^{\xi} \delta_j \tag{3.3}$$

where $\delta = \xi - \eta \in \mathcal{D}$. From (3.1) and (3.3),

$$\max_{\xi \in \Xi} \|y(H_\xi, u, v) - s\|_\infty$$

$$= \max_{\delta \in \mathcal{D}} \max_{p \in \langle rM \rangle} \max_{l \in \langle rM \rangle} \left( \frac{\sum_{i=1}^N (H_\eta^{k+1} - s)^p}{p} + \sum_{j=1}^q (H_j^u^{k+1} - s)^p \right) \tag{3.4}$$

An important step in the derivation involves noting that the order of the three maximizations on the right-hand side of the last equality may be interchanged without altering the value of the overall maximum, so

$$\max_{\xi \in \Xi} \|y(H_\xi, u, v) - s\|_\infty$$

$$= \max_{\delta \in \mathcal{D}} \max_{p \in \langle rM \rangle} \max_{l \in \langle rM \rangle} \left( \alpha_{l}(u, v) \right) + \beta_{l}(u, v) \tag{3.5}$$

where

\[\alpha_l(u, v) := \sum_{i=1}^N (H_{\eta}^{k+1} - s), \quad \beta_{l}(u, v) := \sum_{i=1}^N (H_{\eta}^{k+1} - s).\]

The last expression in (3.4) yields a method for determining the maximum of $\|y(H_\xi, u, v) - s\|_\infty$ with respect to $\xi \in \Xi$ without considering the $2^q$ extreme points of $\Xi$, leading to a considerable saving of computational effort. The problem of minimizing that maximum with respect to $u$ will be converted next into a linear programming problem, using the obvious fact that

$$|a| = \min_{\phi \in \mathbb{R}_+} |a| + \phi = \min_{\phi \in \mathbb{R}_+} \min_{\theta \in \mathbb{R}} |a| + \phi \leq \theta$$

where $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. To facilitate the conversion, a function $f$ will be defined with the property
that $f(u, v, 0, 0)$ is equal to that maximum. In detail, let $f: \mathbb{R}^{mM} \times \mathbb{R}^{N} \times \mathbb{R}^{M} \times \mathbb{R}^{q} \times \mathbb{R}^{r} \rightarrow \mathbb{R}$ be defined by

$$f(u, v, \phi, \psi) = \min \left\{ \theta \in \mathbb{R}: \left( |\alpha_{lp}(u, v) - y_{p}^{k+l}| + \phi_{lp} \right) + \sum_{j=1}^{q} \left( |\beta_{lp}(u, v)| + \psi_{lp} \right) \Delta_{j} \leq \theta, \quad \forall l \in \langle \langle \mathcal{M} \rangle \rangle, \forall p \in \langle \langle \mathcal{R} \rangle \rangle \right\}$$

(3.6)

where $\phi \in \mathbb{R}^{M \times r}$ denotes the array $\{\phi_{lp} \in \mathbb{R}: l \in \langle \langle \mathcal{M} \rangle \rangle, p \in \langle \langle \mathcal{R} \rangle \rangle\}$ and $\psi \in \mathbb{R}^{q} \times \mathbb{R}^{r}$ denotes the array $\{\psi_{lp} \in \mathbb{R}: j \in \langle \langle q \rangle \rangle, l \in \langle \langle \mathcal{M} \rangle \rangle, p \in \langle \langle \mathcal{R} \rangle \rangle\}$. From (3.4) and (3.6),

$$f(u, v, 0, 0) = \max_{\xi \in \Xi} \| y(H_{\xi}, u, v) - s \|_{\infty}$$

(3.7)

so, since $\Delta \geq 0$, it follows from (3.6) that

$$f(u, v, \phi, \psi) \geq f(u, v, 0, 0) = \max_{\xi \in \Xi} \| y(H_{\xi}, u, v) - s \|_{\infty}, \quad \forall \phi \in \mathbb{R}_{\geq}^{M \times r}, \forall \psi \in \mathbb{R}_{\geq}^{q \times \mathbb{R} \times r},$$

(3.8)

where $\mathbb{R}_{\geq}^{M \times r} = \{\phi_{lp} \in \mathbb{R}_{\geq}: l \in \langle \langle \mathcal{M} \rangle \rangle, p \in \langle \langle \mathcal{R} \rangle \rangle\}$, etc. The constraint $y^{L} \leq y(H_{\xi}, u, v) \leq y^{U}$, for all $\xi \in \Xi$, will be transformed next. In view of (3.3) and (3.5),

$$y_{p}^{k+l}(H_{\xi}, u, v) = \sum_{i=1}^{N} \left( H_{n}^{l} y_{p}^{k+i-1} \right) + \sum_{j=1}^{q} \sum_{i=1}^{N} \left( \beta_{lp}(u, v) \right) \Delta_{j} = \alpha_{lp}(u, v) + \sum_{j=1}^{q} \beta_{lp}(u, v) \Delta_{j},$$

(3.9)

It follows that, for all $l \in \langle \langle \mathcal{M} \rangle \rangle$ and for all $p \in \langle \langle \mathcal{R} \rangle \rangle$,

$$\min_{\xi \in \Xi} y_{p}^{k+l}(H_{\xi}, u, v) = \min_{\delta \in \Xi} y_{p}^{k+l}(H_{n+\delta}, u, v) = \alpha_{lp}(u, v) - \sum_{j=1}^{q} \left| \beta_{lp}(u, v) \right| \Delta_{j},$$

$$\max_{\xi \in \Xi} y_{p}^{k+l}(H_{\xi}, u, v) = \max_{\delta \in \Xi} y_{p}^{k+l}(H_{n+\delta}, u, v) = \alpha_{lp}(u, v) + \sum_{j=1}^{q} \left| \beta_{lp}(u, v) \right| \Delta_{j},$$

(3.10)

so the feasible set for $u$ in $\mathbb{P} \mathbb{I}$, namely

$$\mathbb{F} = \{ u \in \mathbb{R}^{nM}: u^{L} \leq u \leq u^{U}, \quad y^{L} \leq y(H_{\xi}, u, v) \leq y^{U}, \forall \xi \in \Xi \},$$

may be written as

$$\mathbb{F} = \{ u \in \mathbb{R}^{nM}: u^{L} \leq u \leq u^{U},$$

$$\left( y_{p}^{L} \right)^{k+l} \leq \alpha_{lp}(u, v) - \sum_{j=1}^{q} \left| \beta_{lp}(u, v) \right| \Delta_{j}, \quad \forall l \in \langle \langle \mathcal{M} \rangle \rangle, \forall p \in \langle \langle \mathcal{R} \rangle \rangle,$$

$$\alpha_{lp}(u, v) + \sum_{j=1}^{q} \left| \beta_{lp}(u, v) \right| \Delta_{j} \leq \left( y_{p}^{U} \right)^{k+l}, \quad \forall l \in \langle \langle \mathcal{M} \rangle \rangle, \forall p \in \langle \langle \mathcal{R} \rangle \rangle \right\}. \quad (3.10)$$

Let $\mathbb{F}_{\phi}: \mathbb{R}^{q} \times \mathbb{R}^{r} \rightarrow 2^{\mathbb{R}^{nM}}$ be defined by

$$\mathbb{F}_{\phi}(\psi) = \left\{ u \in \mathbb{R}^{nM}: u^{L} \leq u \leq u^{U},$$

$$\left( y_{p}^{L} \right)^{k+l} \leq \alpha_{lp}(u, v) - \sum_{j=1}^{q} \left| \beta_{lp}(u, v) \right| \Delta_{j}, \quad \forall l \in \langle \langle \mathcal{M} \rangle \rangle, \forall p \in \langle \langle \mathcal{R} \rangle \rangle,$$

$$\alpha_{lp}(u, v) + \sum_{j=1}^{q} \left| \beta_{lp}(u, v) \right| \Delta_{j} \leq \left( y_{p}^{U} \right)^{k+l}, \quad \forall l \in \langle \langle \mathcal{M} \rangle \rangle, \forall p \in \langle \langle \mathcal{R} \rangle \rangle \right\}. \quad (3.11)$$
Since $\Delta \geq 0$, it follows from (3.10) and (3.11) that
\[ F_\Phi(\psi) \subset F_\Phi(0) = \mathbb{F}, \quad \forall \psi \in R_{\geq}^{q \times M \times r}. \] (3.12)

Hence, from (3.8), (3.12) and (3.7),
\[
\min_{u \in \mathbb{F}(\Phi)} f(u, v, \phi, \psi) \geq \min_{u \in \mathbb{F}(\phi)} f(u, v, 0, 0) \geq \min_{u \in \mathbb{F}} f(u, v, 0, 0)
\]
\[ = \min_{u \in \mathbb{F}} \max_{\xi \in \Xi} \| y(H, u, v) - s \|_\infty, \quad \forall \phi \in R_{\geq}^{M \times r}, \forall \psi \in R_{\geq}^{q \times M \times r}, \]
with all the inequalities holding as equalities when $\phi = 0$ and $\psi = 0$. Consequently,
\[
\min_{\phi \in R_{\geq}^{M \times r}, \psi \in R_{\geq}^{q \times M \times r}, \psi \in \mathbb{F}(\phi)} f(u, v, \phi, \psi) = \min_{u \in \mathbb{F}} \max_{\xi \in \Xi} \| y(H, u, v) - s \|_\infty
\]
and it is easy to verify that the optimization problem on the left is equivalent to that on the right, i.e. is equivalent to $P_1$.

Hence, after making use of (3.6), $P_1$ is equivalent to the following problem:

\[
(P_4) \min_{\theta} \min_{u \in R} \theta,
\]

subject to
\[
\phi \in R_{\geq}^{M \times r}, \quad \psi \in R_{\geq}^{q \times M \times r},
\]
\[
u \in R^{m \times r}, \quad u_L \leq u \leq u_U,
\]
\[
\left( y_p^L \right)^{k+1} \leq \alpha_{ijp}(u, v) - \sum_{j=1}^q \left( | \beta_{ijp}(u, v) | + \psi_{ijp} \right) \Delta_j,
\]
\[
\alpha_{ijp}(u, v) + \sum_{j=1}^q \left( | \beta_{ijp}(u, v) | + \psi_{ijp} \right) \Delta_j \leq \left( y_p^U \right)^{k+1}, \quad \forall l \in \angle \langle M \rangle, \forall p \in \angle \langle r \rangle,
\]
\[
\left( | \alpha_{ijp}(u, v) - s_p^{k+1} | + \phi_{ijp} \right) + \sum_{j=1}^q \left( | \beta_{ijp}(u, v) | + \psi_{ijp} \right) \Delta_j \leq \theta, \quad \forall l \in \angle \langle M \rangle, \forall p \in \angle \langle r \rangle.
\]

Now $( | \alpha_{ijp}(u, v) - s_p^{k+1} | + \phi_{ijp} )$ and the constraint $\phi_{ijp} \in R_{\geq}$ in $P_4$ may be replaced by $\Phi_{ijp}$ and the constraints $\Phi_{ijp} \in R_{\geq}$ by $\alpha_{ijp}(u, v) - s_p^{k+1}$, i.e. by $\Phi_{ijp} \in R_{\geq}$, $-\Phi_{ijp} \leq \alpha_{ijp}(u, v) - s_p^{k+1} \leq \Phi_{ijp}$. Similarly $( | \beta_{ijp}(u, v) | + \psi_{ijp} )$ and the constraint $\psi_{ijp} \in R_{\geq}$ may be replaced by $\Psi_{ijp}$ and the constraints $\Psi_{ijp} \in R_{\geq}$, $-\Psi_{ijp} \leq \beta_{ijp}(u, v) \leq \Psi_{ijp}$. Then, after substituting for $\alpha_{ijp}$ and $\beta_{ijp}$ from (3.5), it may be seen easily that $P_4$ is equivalent to

\[
(P_5) \min_{\theta} \theta,
\]

subject to
\[
\theta, \Phi_{ijp}, \Psi_{ijp} \in R, \quad \forall j \in \angle \langle q \rangle, \forall l \in \angle \langle M \rangle, \forall p \in \angle \langle r \rangle,
\]
\[
u \in R^{m \times r}, \quad u_L \leq u \leq u_U,
\]
\[
\Phi_{ijp} + \sum_{j=1}^q \Psi_{ijp} \Delta_j \leq \theta, \quad \forall l \in \angle \langle M \rangle, \forall p \in \angle \langle r \rangle,
\]
\[
-\Phi_{ijp} \leq \sum_{i=1}^N \left( H_n u_{k+1}^{k+i-} \right)_p - s_p^{k+1} \leq \Phi_{ijp}, \quad \forall l \in \angle \langle M \rangle, \forall p \in \angle \langle r \rangle,
\]
\[
-\Psi_{ijp} \leq \sum_{i=1}^N \left( H_n u_{k+1}^{k+i-} \right)_p \leq \Psi_{ijp}, \quad \forall j \in \angle \langle q \rangle, \forall l \in \angle \langle M \rangle, \forall p \in \angle \langle r \rangle,
\]
\[
\left( y_p^L \right)^{k+1} \leq \left( \sum_{i=1}^N H_n u_{k+1}^{k+i-} \right)_p - \sum_{j=1}^q \Psi_{ijp} \Delta_j, \quad \forall l \in \angle \langle M \rangle, \forall p \in \angle \langle r \rangle,
\]
\[
\left( \sum_{i=1}^N H_n u_{k+1}^{k+i-} \right)_p + \sum_{j=1}^q \Psi_{ijp} \Delta_j \leq \left( y_p^U \right)^{k+1}, \quad \forall l \in \angle \langle M \rangle, \forall p \in \angle \langle r \rangle.
\]
Since $P4$ is equivalent to both $P1$ and $P5$, linear programming problem $P5$ is equivalent to min–max problem $P1$. It is easy to check that any $u$ which is minimizing for $P5$ is also minimizing for $P1$, so use of $P5$ to solve $P1$ is an alternative to the use of $P3$ to solve $P1$.

4. Concluding remarks

The advantage of solving min–max problem $P1$ using the linear program $P5$ derived in Section 3 instead of using the linear program $P3$ (of the type derived by Campo and Morari) is that $P5$ has $\kappa_5 = (2q + 5)rM$ constraints, which is a significantly smaller number of constraints than the number $\kappa_3 = 2(m + r2q + 1)M$ of constraints for $P3$. The reduction in the number of constraints facilitates the application of robust model predictive control to plants with non-small $q$, i.e. for which the impulse response may range over a polytope of non-small dimension – hence the potential advantage of the approach developed here.

The case with $\| y(H_\xi, u, v) - s \|_1$ instead of $\| y(H_\xi, u, v) - s \|_\infty$ could be treated in the same way by incorporating the fact that

$$\| y(H_\xi, u, v) - s \|_1 = \max_{i \in \{0,1\}^{2^M}} v'_i \left( y(H_\xi, u, v) - s \right),$$

where the $v'_i$ are the rows of any $2^M \times rM$ matrix $V$ which has all possible arrangements of $+1$ and $-1$ in its rows, but better methods are being developed for that problem, e.g. [3].

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References