

Weakly Coupled Oscillators

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 (Dated: May 27, 2005)

PACS numbers:

Introduction

Practically any physical, chemical, or biological system can exhibit rhythmic oscillatory activity, at least when the conditions are right. Winfree (2001) reviews the ubiquity of oscillations in nature, ranging from auto-catalytic chemical reactions to pacemaker cells in the heart, to animal gates, and to circadian rhythms. When coupled, even weakly, oscillators interact via adjustment of their phases, i.e., their timing, often leading to synchronization. In this chapter we review the most important concepts needed to study and understand the dynamics of coupled oscillators.

From mathematical point of view, an *oscillator* is a dynamical system

$$\dot{x} = f(x), \quad x \in \mathbb{R}^m, \quad (1)$$

having a limit cycle attractor – periodic orbit $\gamma \subset \mathbb{R}^m$. Its *period* is the minimal $T > 0$ such that

$$\gamma(t) = \gamma(t + T) \quad \text{for any } t,$$

and its *frequency* is $\Omega = 2\pi/T$. Let $x(0) = x_0 \in \gamma$ be an arbitrary point on the attractor, then the state of the system, $x(t)$, is uniquely defined by its phase $\vartheta \in \mathbb{S}^1$ relative to x_0 , where \mathbb{S}^1 is the unit circle.

Throughout this chapter we assume that the periodic orbit γ is exponentially stable, which implies normal hyperbolicity. In this case, there is a continuous transformation $\Theta : U \rightarrow \mathbb{S}^1$ defined in a neighborhood $U \supset \gamma$ such that $\vartheta(t) = \Theta(x(t))$ for any trajectory in U . That is, Θ maps solutions of (1) to solutions of

$$\dot{\vartheta} = \Omega. \quad (2)$$

Such a transformation removes the amplitude but saves the phase of oscillation.

Accordingly, there is a continuous transformation that maps solutions of the weakly coupled network of n oscillators

$$\dot{x}_i = f_i(x_i) + \varepsilon g_i(x_1, \dots, x_n, \varepsilon), \quad \varepsilon \ll 1, \quad (3)$$

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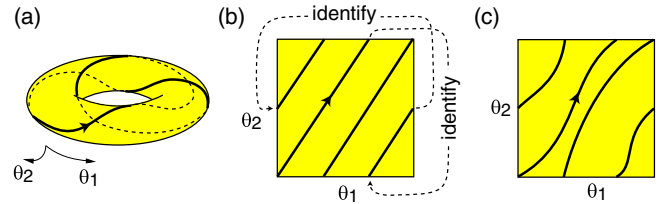


FIG. 1: 2-torus and its representation on the square.

onto solutions of the phase system

$$\dot{\vartheta}_i = \Omega_i + \varepsilon h_i(\vartheta_1, \dots, \vartheta_n, \varepsilon), \quad \vartheta_i \in \mathbb{S}^1, \quad (4)$$

which is easier to study the collective properties of (3).

The oscillators are said to be *frequency locked* when (4) has a stable periodic orbit $\vartheta(t) = (\vartheta_1(t), \dots, \vartheta_n(t))$ on the n -torus \mathbb{T}^n , as in Fig. 1a. The *rotation vector* or *winding ratio* of the orbit is the set of integers $q_1 : q_2 : \dots : q_n$ such that ϑ_1 makes q_1 rotations while ϑ_2 makes q_2 rotations, etc., as in the 2:3 frequency locking in Fig. 1a. The oscillators are *entrained* when they are 1:1:⋯:1 frequency locked. The oscillators are *phase locked* when there is an $(n-1) \times n$ integer matrix K having linearly independent rows such that $K\vartheta(t) = \text{const}$. For example, the two oscillators in Fig. 1b are phase locked with $K = (2, 3)$, while those in Fig. 1c are not. The oscillators are *synchronized* when they are entrained and phase locked. Synchronization is *in-phase* when $\vartheta_1(t) = \dots = \vartheta_n(t)$ and *out-of-phase* otherwise. Two oscillators are said to be synchronized *anti-phase* when $\vartheta_1(t) - \vartheta_2(t) = \pi$. Frequency locking without phase locking, as in Fig. 1c, is called *phase trapping*. The relationship between all these definitions is depicted in Fig. 2.

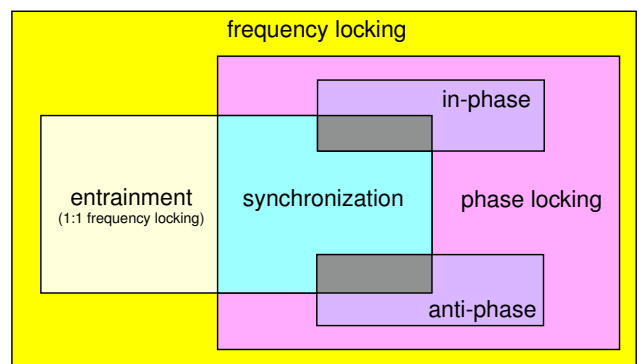


FIG. 2: Various degrees of locking of oscillators.

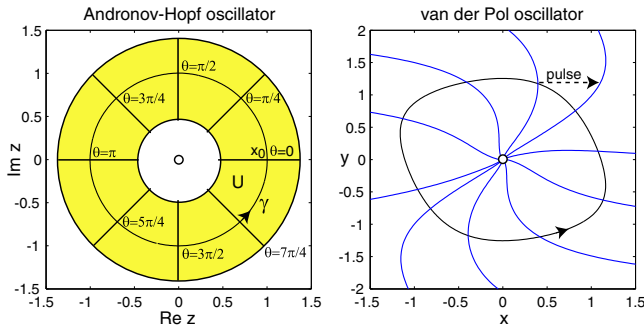


FIG. 3: Isochrons of Andronov-Hopf oscillator ($\dot{z} = (1+i)z - z|z|^2$, $z \in \mathbb{C}$) and van der Pol oscillator ($\dot{x} = x - x^3 - y$, $\dot{y} = x$).

Phase resetting

An exponentially stable periodic orbit is a normally hyperbolic invariant manifold, hence its sufficiently small neighborhood, U , is invariantly foliated by stable submanifolds (Guckenheimer 1975) illustrated in Fig. 3. The manifolds represent points having equal phases, and for this reason, they are called *isochrons* (from Greek *iso* meaning *equal* and *chronos* meaning *time*).

The geometry of isochrons determines how the oscillators react to perturbations. For example, the pulse in Fig. 3, right, moves the trajectory from one isochron to another, thereby changing its phase. The magnitude of the phase shift depends on the amplitude and the exact timing of the stimulus relative to the phase of oscillation ϑ . Stimulating the oscillator at different phases, one can measure the *phase transition curve* (Winfree 1980)

$$\vartheta_{\text{new}} = \text{PTC}(\vartheta_{\text{old}})$$

and the *phase resetting curve*

$$\text{PRC}(\vartheta) = \text{PTC}(\vartheta) - \vartheta \quad (\text{shift} = \text{new phase} - \text{old phase}).$$

Positive (negative) values of the PRC correspond to phase advances (delays). PRCs are convenient when the phase shifts are small, so that they can be magnified and clearly seen, as in Fig. 4. PTCs are convenient when the phase shifts are large and comparable with the period of oscillation.

In Fig. 5 we depict phase portraits of the Andronov-Hopf oscillator receiving pulses of magnitude 0.5 (left) and 1.5 (right). Notice the drastic difference between the corresponding PRCs or PTCs. Winfree (2001) distinguishes two cases:

- *Type 1 (weak) resetting* results in continuous PRCs and PTCs with mean slope 1.
- *Type 0 (strong) resetting* results in discontinuous PRCs and PTCs with mean slope 0.

The discontinuity of Type 0 PRC in Fig. 5 is a topological property that cannot be removed by reallocating

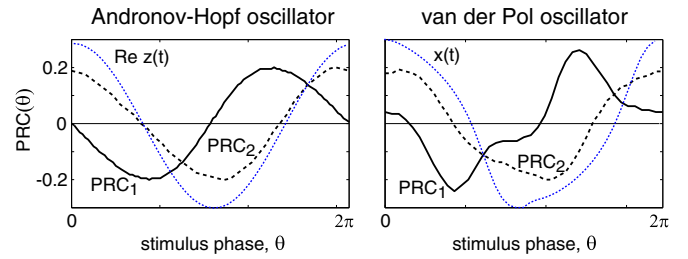


FIG. 4: Examples of phase response curves (PRC) of the oscillators in Fig. 3. $\text{PRC}_1(\vartheta)$ and $\text{PRC}_2(\vartheta)$ correspond to horizontal (along the first variable) and vertical (along the second variable) pulses with amplitudes 0.2. An example of oscillation is plotted as a dotted curve in each subplot (not to scale).

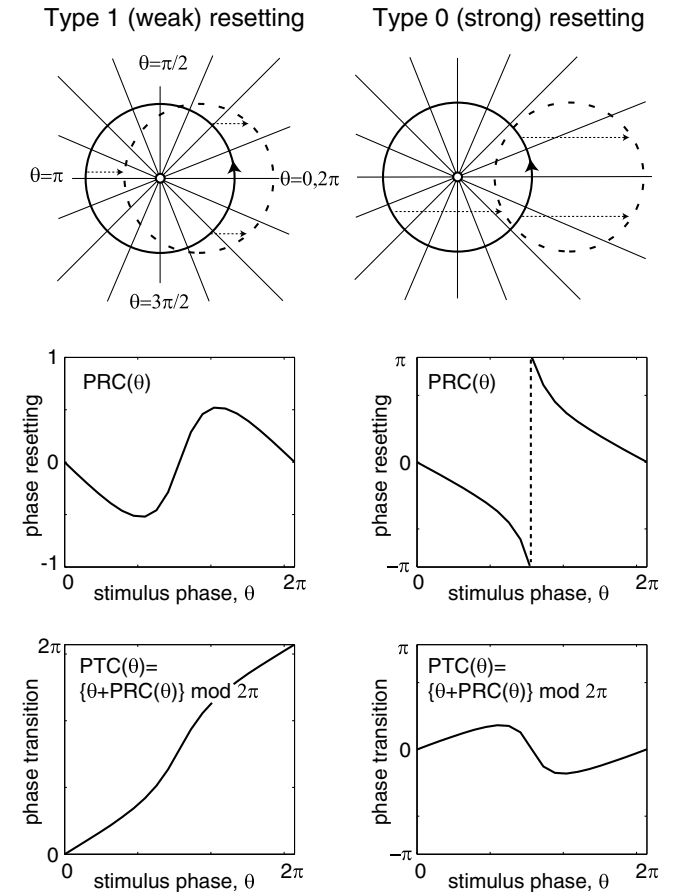


FIG. 5: Types of phase resetting of the Andronov-Hopf oscillator in Fig. 3.

the initial point x_0 that corresponds to zero phase. The discontinuity stems from the fact that the shifted image of the limit cycle (dashed circle) goes beyond the central equilibrium at which the phase is not defined.

The stroboscopic mapping of \mathbb{S}^1 to itself, called *Poincare phase map*,

$$\vartheta_{k+1} = \text{PTC}(\vartheta_k), \quad (5)$$

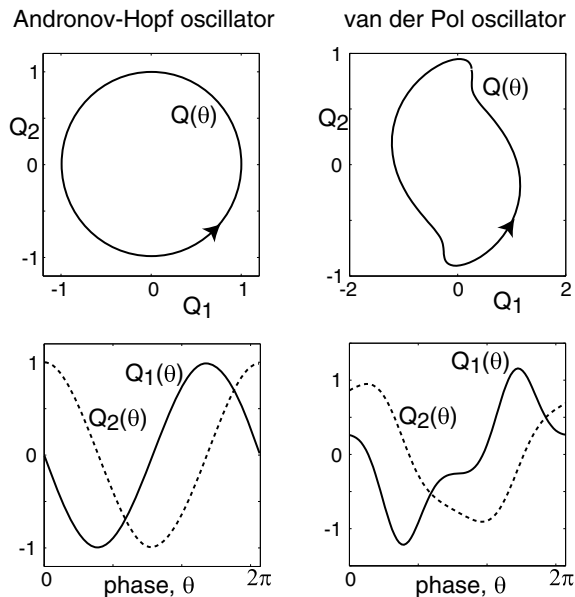


FIG. 6: Solutions $Q = (Q_1, Q_2)$ to the adjoint problem (7) for oscillators in Fig. 3.

describes the response of an oscillator to a T -periodic pulse train. Here, ϑ_k denotes the phase of oscillation when the k th input pulse arrives. Its fixed points correspond to synchronized solutions, and its periodic orbits correspond to phase locked states.

Weak Coupling

Now consider dynamical systems of the form

$$\dot{x} = f(x) + \varepsilon s(t), \quad (6)$$

describing periodic oscillators, $\dot{x} = f(x)$, forced by a weak time-dependent input $\varepsilon s(t)$, e.g., from other oscillators in a network. Let $\Theta(x)$ denote the phase of oscillation at point $x \in U$, so that the map $\Theta : U \rightarrow \mathbb{S}^1$ is constant along each isochron. This mapping transforms (6) into the phase model

$$\dot{\vartheta} = \Omega + \varepsilon Q(\vartheta) \cdot s(t),$$

with function $Q(\vartheta)$, illustrated in Fig. 6, satisfying three equivalent conditions:

- Winfree: $Q(\vartheta)$ is normalized PRC to infinitesimal pulsed perturbations.
- Kuramoto: $Q(\vartheta) = \text{grad } \Theta(x)$.
- Malkin: Q is the solution to the adjoint problem

$$\dot{Q} = -\{Df(\gamma(t))\}^\top Q, \quad (7)$$

with the normalization $Q(t) \cdot f(\gamma(t)) = \Omega$ for any t .

Function $Q(\vartheta)$ can be found analytically in a few simple cases:

- A nonlinear phase oscillator $\dot{x} = f(x)$ with $x \in \mathbb{S}^1$ and $f > 0$ has $Q(\vartheta) = \Omega/f(\gamma(\vartheta))$.
- A system near saddle-node on invariant circle bifurcation has $Q(\vartheta)$ proportional to $1 - \cos \vartheta$.
- A system near supercritical Andronov-Hopf bifurcation has $Q(\vartheta)$ proportional to $\sin(\vartheta - \psi)$, where $\psi \in \mathbb{S}^1$ is a constant phase shift.

Other interesting cases, including homoclinic, relaxation, and bursting oscillators are considered by Izhikevich (2005).

Treating $s(t)$ in (6) as the input from the network, we can transform weakly coupled oscillators

$$\dot{x}_i = f_i(x_i) + \varepsilon \overbrace{\sum_{j=1}^n g_{ij}(x_i, x_j)}^{s_i(t)}, \quad x_i \in \mathbb{R}^m, \quad (8)$$

to the phase model

$$\dot{\vartheta}_i = \Omega_i + \varepsilon Q_i(\vartheta_i) \cdot \overbrace{\sum_{j=1}^n g_{ij}(x_i(\vartheta_i), x_j(\vartheta_j))}^{s_i(t)}, \quad (9)$$

having the form (4) with $h_i = Q_i \sum g_{ij}$, or the form

$$\dot{\vartheta}_i = \Omega_i + \varepsilon \sum_{j=1}^n h_{ij}(\vartheta_i, \vartheta_j),$$

where $h_{ij} = Q_i g_{ij}$. Introducing phase deviation variables $\vartheta_i = \Omega_i t + \varphi_i$, we transform this system into the form

$$\dot{\varphi}_i = \varepsilon \sum_{j=1}^n h_{ij}(\Omega_i t + \varphi_i, \Omega_j t + \varphi_j),$$

which can be averaged to

$$\dot{\varphi}_i = \varepsilon \sum_{j=1}^n H_{ij}(\varphi_i - \varphi_j), \quad (10)$$

with the functions

$$H_{ij}(\chi) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T h_{ij}(\Omega_i t, \Omega_j t - \chi) dt \quad (11)$$

describing the interaction between oscillators. To summarize, we transformed weakly coupled system (8) into the phase model (10) with H given by (11) and each Q being the solution to the adjoint problem (7). This constitutes the Malkin theorem for weakly coupled oscillators (Hoppensteadt and Izhikevich 1997, Theorem 9.2).

Existence of one equilibrium of the phase model (10) implies the existence of the entire circular family of equilibria, since translation of all φ_i by a constant phase shift

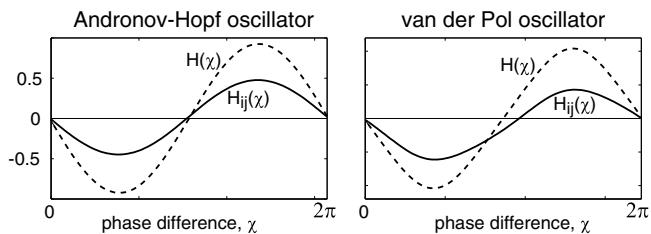


FIG. 7: *Solid curves*: Functions $H_{ij}(\chi)$ defined by (11) corresponding to the gap-junction input $g(x_i, x_j) = (x_{j1} - x_{i1}, 0)$. *Dashed curves*: Functions $H(\chi) = H_{ji}(\chi) - H_{ij}(-\chi)$. Parameters as in Fig. 3.

does not change the phase differences $\varphi_i - \varphi_j$ and hence the form of (10). This family corresponds to a limit cycle of (8), on which all oscillators have equal frequencies and constant phase shifts, i.e., they are synchronized, possibly out-of-phase.

We say that two oscillators, i and j , have *resonant* (or commensurable) frequencies when the ratio Ω_i/Ω_j is a rational number, e.g., it is p/q for some integer p and q . They are non-resonant when the ratio is an irrational number. In this case the function H_{ij} defined above is constant regardless of the details of the oscillatory dynamics or the details of the coupling. That is, dynamic of two coupled non-resonant oscillators is described by an uncoupled phase model. Apparently, such oscillators do not interact; that is, the phase of one of them cannot change the phase of the other one even on the long time scale of order $1/\varepsilon$.

Synchronization

Consider (8) with $n = 2$, describing two mutually coupled oscillators. Let us introduce “slow” time $\tau = \varepsilon t$ and rewrite the corresponding phase model (10) in the form

$$\begin{aligned}\varphi_1' &= \omega_1 + H_{12}(\varphi_1 - \varphi_2) \\ \varphi_2' &= \omega_2 + H_{21}(\varphi_2 - \varphi_1),\end{aligned}$$

where $' = d/d\tau$ and $\omega_i = H_{ii}(0)$ is the frequency deviation from the natural oscillation, $i = 1, 2$. Let $\chi = \varphi_2 - \varphi_1$ denote the phase difference between the oscillators, then

$$\chi' = \omega + H(\chi), \quad (12)$$

where

$$\omega = \omega_2 - \omega_1 \quad \text{and} \quad H(\chi) = H_{21}(\chi) - H_{12}(-\chi),$$

is the frequency mismatch and the anti-symmetric part of the coupling, respectively, illustrated in Fig. 7, dashed curves. A stable equilibrium of (12) corresponds to a stable limit cycle of the phase model.

All equilibria of (12) are solutions to $H(\chi) = -\omega$, and they are intersections of the horizontal line $-\omega$ with the graph of H . They are stable if the slope of the graph is

negative at the intersection. If oscillators are identical, then $H(\chi)$ is an odd function (i.e., $H(-\chi) = -H(\chi)$), and $\chi = 0$ and $\chi = \pi$ are always equilibria, possibly unstable, corresponding to the in-phase and anti-phase synchronized solutions. The in-phase synchronization of gap-junction coupled oscillators in Fig. 7 is stable because the slope of H (dashed curves) is negative at $\chi = 0$. The max and min values of the function H determine the tolerance of the network to the frequency mismatch ω , since there are no equilibria outside this range.

Now consider a network of $n > 2$ weakly coupled oscillators (8). To determine the existence and stability of synchronized states in the network, we need to study equilibria of the corresponding phase model (10). Vector $\phi = (\phi_1, \dots, \phi_n)$ is an equilibrium of (10) when

$$0 = \omega_i + \sum_{j \neq i}^n H_{ij}(\phi_i - \phi_j) \quad (\text{for all } i). \quad (13)$$

It is stable when all eigenvalues of the linearization matrix (Jacobian) at ϕ have negative real parts, except one zero eigenvalue corresponding to the eigenvector along the circular family of equilibria (ϕ plus a phase shift is a solution of (13) too since the phase shifts $\phi_j - \phi_i$ are not affected).

In general, determining the stability of equilibria is a difficult problem. Ermentrout (1992) found a simple sufficient condition. If

- $a_{ij} = H'_{ij}(\phi_i - \phi_j) \leq 0$, and
- the directed graph defined by the matrix $a = (a_{ij})$ is connected, (i.e., each oscillator is influenced, possibly indirectly, by every other oscillator),

then the equilibrium ϕ is neutrally stable, and the corresponding limit cycle $x(t + \phi)$ of (8) is asymptotically stable.

Another sufficient condition was found by Hoppensteadt and Izhikevich (1997). If system (10) satisfies

- $\omega_1 = \dots = \omega_n = \omega$ (identical frequencies)
- $H_{ij}(-\chi) = -H_{ji}(\chi)$ (pair-wise odd coupling)

for all i and j , then the network dynamics converge to a limit cycle. On the cycle, all oscillators have equal frequencies $1 + \varepsilon\omega$ and constant phase deviations.

The proof follows from the observation that (10) is a gradient system in the rotating coordinates $\varphi = \omega\tau + \phi$ with the energy function

$$E(\phi) = -\frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n R_{ij}(\phi_i - \phi_j),$$

where

$$R_{ij}(\chi) = \int_0^\chi H_{ij}(s) ds.$$

One can check that $dE(\phi)/d\tau = -\sum(\phi_i')^2 \leq 0$ along the trajectories of (12) with equality only at equilibria.

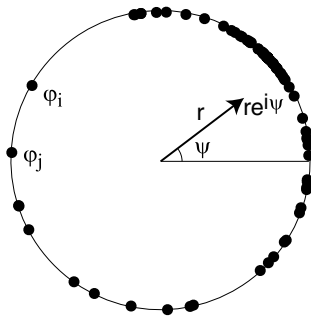


FIG. 8: Kuramoto synchronization index (15) describes the degree of coherence in the network (14).

Mean-Field Approximations

Let us represent the phase model (10) in the form

$$\varphi'_i = \omega_i + \sum_{j \neq i}^n H_{ij}(\varphi_i - \varphi_j)$$

where $' = d/d\tau$, $\tau \varepsilon t$ is the slow time, and $\omega_i = H_{ii}(0)$ are random frequency deviations. Collective dynamics of this system can be analyzed in the limit $n \rightarrow \infty$. We illustrate the theory using the special case, $H(\chi) = -\sin \chi$, known as the Kuramoto (1984) model

$$\varphi'_i = \omega_i + \frac{K}{n} \sum_{j=1}^n \sin(\varphi_j - \varphi_i), \quad \varphi_i \in [0, 2\pi], \quad (14)$$

where $K > 0$ is the coupling strength and the factor $1/n$ ensures that the model behaves well as $n \rightarrow \infty$. The complex-valued sum of all phases,

$$r e^{i\psi} = \frac{1}{n} \sum_{j=1}^n e^{i\varphi_j}, \quad (\text{Kuramoto synchronization index}) \quad (15)$$

describes the degree of synchronization in the network. Apparently, the in-phase synchronized state $\varphi_1 = \dots = \varphi_n$ corresponds to $r = 1$ with ψ being the population phase. In contrast, the *incoherent* state with all φ_i having different values randomly distributed on the unit circle, corresponds to $r \approx 0$. Intermediate values of r correspond to a *partially synchronized* or *coherent* state, depicted in Fig. 8. Some phases are synchronized forming a cluster, while others roam around the circle.

Multiplying both sides of (15) by $e^{-i\varphi_i}$ and considering only the imaginary parts, we can rewrite (14) in the equivalent form

$$\varphi'_i = \omega_i + K r \sin(\psi - \varphi_i)$$

that emphasizes the mean-field character of interactions between the oscillators: They all are pulled into the synchronized cluster ($\varphi_i \rightarrow \psi$) with the effective strength proportional to the cluster size r . This pull is offset by the random frequency deviations ω_i that pull away from the cluster.

Let us assume that omegas are distributed randomly around 0 with a symmetrical probability density function $g(\omega)$, e.g., Gaussian. Kuramoto has shown that in the limit $n \rightarrow \infty$, the cluster size r obeys the self-consistency equation

$$r = rK \int_{-\pi/2}^{+\pi/2} g(Kr \sin \varphi) \cos^2 \varphi d\varphi. \quad (16)$$

Notice that $r = 0$, corresponding to the incoherent state, is always a solution of this equation. When the coupling strength K is greater than a certain critical value,

$$K_c = \frac{2}{\pi g(0)},$$

an additional, nontrivial solution $r > 0$ appears, which corresponds to a partially synchronized state. Expanding g in a Taylor series, one get the scaling $r = \sqrt{16(K - K_c)/(-g''(0)\pi K_c^4)}$. Thus, the stronger the coupling K relative to the random distribution of frequencies, the more oscillators synchronize into a coherent cluster. The issue of stability of incoherent and partially synchronized states is discussed by Strogatz (2000). Other generalizations of the Kuramoto model are reviewed by Acebron et al. (2005).

Further Reading

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